

Common Fixed Point Theorems for Four Selfmaps of a Complete S-Metric space

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Abstract

Suppose (X, S) is a S- metric space and P, T, I and J are selfmaps of X . If these four maps and the space X satisfy certain conditions, we shall prove that they have a unique common fixed point in this paper. As a consequence we deduce a common fixed point theorem for four selfmaps of a complete S- metric space. Further, we show that a common fixed point theorem for four selfmaps of a metric space proved by Brain Fisher ([5]) follows as a particular case of the theorem.

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Key Words: S-metric space; Associated sequence; Fixed point theorem

1. Introduction and Preliminaries:

One of the fields that many researchers are actively studying is fixed point theory, particularly in analysis. Fixed point theory can be applied in a variety of fields, including biology, computer science, engineering, economics, etc. The Banach Contraction Principle is known as one of the most important findings in fixed point theory.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gähler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2- metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [7], [8] and [9]. Sedghi et al. [10] modified the concepts of D- metric space and introduced the concepts of D*- metric space also proved a common fixed point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they gives following concepts of S- metric space.

Definition 1.1 ([11]): Let X be a non-empty set. An S-metric space on X is a function $S: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$

- (i) $S(x, y, z) \geq 0$
- (ii) $S(x, y, z) = 0$ if and only if $x = y = z$.
- (iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S-metric space**.

Immediate examples of such S-metric spaces are:

Example 1.2: Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each $x, y, z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X .

Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S- metric on X . If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$\begin{aligned}
S(x, y, z) &= d(x, y) + d(y, z) + d(z, x) \\
&\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x) \\
&= S(x, x, a) + S(y, y, a) + S(z, z, a)
\end{aligned}$$

Example 1. 4: Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S-metric on X .

Remark 1. 5: it is easy to see that every D^* -metric is S-metric, but in general the converse is not true, see the following example.

Example 1. 6: Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X , but it is not D^* -metric because it is not symmetric.

Lemma 1. 7: In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S-metric, we get

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x) \dots (1)$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y) \dots (2)$$

Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition 1.8: Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X; S(x, y, y) < r\}$$

$$B_S[x, r] = \{y \in X; S(x, y, y) \leq r\}$$

For example, Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Therefore $B_S(1, 2) = \{y \in \mathbb{R}; S(y, y, 1) < 2\}$

$$= \{y \in \mathbb{R}; |y - 1| < 1\} = (0, 2).$$

Definition 1.9: Let (X, S) be an S-metric space and $A \subset X$.

- 1) If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X
- 2) If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S-bounded**.
- 3) A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n \rightarrow \infty} x_n = x$
- 4) A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$
- 5) The S-metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.
- 6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).
- 7) If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S-metric space.

Lemma 1. 10 ([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball $B_S(x, r)$ is an open subset of X .

Lemma 1. 11 ([11]): Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x , then x is unique.

Lemma 1. 12 ([11]): Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x , then $\{x_n\}$ is a Cauchy sequence.

Lemma 1. 13 ([11]): Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 1. 14: Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$ is an S-metric on X
2. $x_n \rightarrow x$ in (X, d) if and only if $x_n \rightarrow x$ in (X, S_d)
3. $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)
4. (X, d) is complete if and only if (X, S_d) is complete

Proof: (1) See [Example (3), Page 260]

(2) $x_n \rightarrow x$ in (X, d) if and only if $d(x_n, x) \rightarrow 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \rightarrow 0$ that is, $x_n \rightarrow x$ in (X, S_d)

(3) $\{x_n\}$ is a Cauchy in (X, d) if and only if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \rightarrow 0$, $n, m \rightarrow \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d)

(4) It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X , we denote $T(x)$ by Tx .

If P and Q are selfmaps of a set X , then any $z \in X$ such that $Pz = Qz = z$ is called a **common fixed point** of P and Q . Two selfmaps P and Q of X are said to be **commutative** if $PQ = QP$ where PQ is their composition $P \circ Q$ defined by $(P \circ Q)x = PQx$ for all $x \in X$.

Definition 1.15: Suppose P and Q are selfmaps of a S-metric space (X, S) satisfying the condition $Q(X) \subseteq P(X)$. Then for any $x_0 \in X$, $Qx_0 \in Q(X)$ and hence $Qx_0 \in P(X)$, so that there is a $x_1 \in X$ with $Qx_0 = Px_1$, since $Q(X) \subseteq P(X)$. Now $Qx_1 \in Q(X)$ and hence there is a $x_2 \in X$ with $Qx_1 \in Q(X) \subseteq P(X)$ so that $Qx_1 = Px_2$. Again $Qx_2 \in Q(X)$ and hence $Qx_2 \in P(X)$ with $Qx_2 = Px_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$. We shall call this sequence as an **associated sequence of x_0 relative to the two selfmaps P and Q** . It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps P and Q .

Let P and Q are selfmaps of a S-metric space (X, S) such that $Q(X) \subseteq P(X)$. For any $x_0 \in X$, if $\{x_n\}$ is a sequence in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps P and Q .

Definition 1.16: A function $\emptyset: [0, \infty) \rightarrow [0, \infty)$ is said to be a **contractive modulus**, if $\emptyset(0) = 0$ and $\emptyset(t) < t$ for $t > 0$.

Definition 1.17: A real valued function $\emptyset$ defined on $X \subseteq \mathbb{R}$ is said to be **upper semi continuous**, if $\limsup_{n \rightarrow \infty} \emptyset(t_n) \leq \emptyset(t)$ for every sequence $\{t_n\}$ in X with $t_n \rightarrow t$ as $n \rightarrow \infty$.

Definition 1.18: If P and Q are selfmaps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = t$, we have $\lim_{n \rightarrow \infty} S(PQx_n, QPx_n, QPx_n) = 0$, then we say that P and Q are **compatible**.

The Main Results:

2. 1 Theorem: Let P, T, I and J be selfmaps of a S- metric space (X, S) satisfying the conditions

(i) $P(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$

(ii) $S(Px, Ty, Ty) \leq \emptyset(\mu(x, y))$ for all $x, y \in X$

where $\emptyset$ is an upper semi continuous and contractive modulus

and

(ii)' $\mu(x, y) = \max \{S(Ix, Jy, Jy), S(Ix, Px, Px), S(Jy, Ty, Ty)\}$ for all $x, y \in X$

(iii) one of P, T, I and J is continuous

and

(iv) the pairs (P, I) and (T, J) are compatible

Further, if

(iv) there exists a point $x_0 \in X$ and an associated sequence $\{x_n\}$ relative to the four selfmaps such that $Px_0, Tx_1, Px_2, Tx_3, \dots, Px_{2n}, Tx_{2n+1}, \dots$ converge to some $z \in X$,

Then P, T, I and J have a unique common fixed point $z \in X$. Also z is the unique common fixed point for the pair P and I; and for the pair T and J.

Before we give proof of theorem, we establish some lemmas.

1.1. 1 Lemma: Suppose P, T, I and J are selfmaps of a S-metric space satisfying the conditions (i), (ii), and (v) of theorem 2.1 and pair (P, I) is compatible. Then

- (a) $\lim_{n \rightarrow \infty} \mu(Ix_{2n}, x_{2n+1}) = S(Iz, z, z)$ whenever I is continuous
and
(b) $\lim_{n \rightarrow \infty} \mu(Px_{2n}, x_{2n+1}) = S(Pz, z, z)$ whenever P is continuous

Proof: In view of (v), the sequences $\{Px_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to $z \in X$ and since $Px_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ we have

(2. 1. 2) $Px_{2n}, Tx_{2n+1}, Ix_{2n}, Jx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$.

(a) If I is continuous, then we have

(2.1.3) $I^2x_{2n} \rightarrow Iz, IPx_{2n} \rightarrow Iz$ as $n \rightarrow \infty$

Also compatible of the pair (P, I) implies

(2.1.4) $\lim_{n \rightarrow \infty} S(PIx_{2n}, IPx_{2n}, IPx_{2n}) = 0$

From (2. 1. 3) and (2. 1. 4), we get that

(2.1.5) $PIx_{2n} \rightarrow Iz$ as $n \rightarrow \infty$.

Now, from (ii)', we have

(2. 1. 6) $\mu(Ix_{2n}, x_{2n+1}) = \max \{S(I^2x_{2n}, Jx_{2n+1}, Jx_{2n+1}), S(I^2x_{2n}, SIx_{2n}, SIx_{2n}), S(Jx_{2n}, Tx_{2n+1}, Tx_{2n+1})\}$

Letting n to ∞ in (2. 1. 6) and using the continuity of S, (2. 1. 2), (2. 1. 3) and (2. 1. 5) we get

$$\lim_{n \rightarrow \infty} \mu(Ix_{2n}, x_{2n+1}) = \max \{S(Iz, z, z), S(Iz, z, z), S(z, z, z)\} = S(Iz, z, z).$$

This proves part (a) of the lemma.

(b) If P is continuous, then, by (2. 1. 2), we have

(2. 1. 7) $P^2x_{2n} \rightarrow Pz, PIPx_{2n} \rightarrow Pz$ as $n \rightarrow \infty$.

and therefore in view of (2. 1. 4) we get

(2. 1. 8) $IPx_{2n} \rightarrow Pz$ as $n \rightarrow \infty$.

Now we have

(2. 1. 9) $\mu(Px_{2n}, x_{2n+1}) = \max \{S(IPx_{2n}, Jx_{2n+1}, Jx_{2n+1}), S(IPx_{2n}, P^2x_{2n}, P^2x_{2n}), S(Jx_{2n+1}, Tx_{2n+1}, Tx_{2n+1})\}$

In which on letting n to ∞ and using (2. 1. 2), (2. 1. 7), (2. 1. 8) and continuity of S we get $\lim_{n \rightarrow \infty} \mu(Px_{2n}, x_{2n+1}) = \max \{S(Pz, z, z), S(Pz, z, z), S(z, z, z)\}$

$= S(Pz, z, z)$. proving part (b) of the lemma.

(2. 1. 10) Lemma: Suppose P, T, I and J are selfmaps of a S- metric space(X, S) satisfying the conditions (i), (ii), and (v) of theorem 2.1 and pair (T, J) is compatible. Then

- (a) $\lim_{n \rightarrow \infty} \mu(x_{2n}, Jx_{2n+1}) = S(z, Jz, Jz)$ whenever J is continuous
and
(b) $\lim_{n \rightarrow \infty} \mu(x_{2n}, Tx_{2n+1}) = S(z, Tz, Tz)$ whenever T is continuous

Proof: (a) If I is continuous, then we have, in view of (2. 1. 2) that

(2.1.11) $J^2x_{2n} \rightarrow Jz, JT x_{2n+1} \rightarrow Jz$ as $n \rightarrow \infty$

Also the pair (T, J) is compatible implies that

(2.1.12) $\lim_{n \rightarrow \infty} S(TJx_{2n+1}, JT x_{2n+1}, JT x_{2n+1}) = 0$

From (2. 1. 11) and (2. 1. 12), we get that

(2.1.13) $TJx_{2n+1} \rightarrow Jz$ as $n \rightarrow \infty$.

Now, from (ii)', we have

(2. 1. 14) $\mu(x_{2n}, Jx_{2n+1}) = \max \{S(Ix_{2n}, J^2x_{2n+1}, J^2x_{2n+1}), S(Ix_{2n}, Px_{2n}, Px_{2n}), S(Jx_{2n+1}, TJx_{2n+1}, TJx_{2n+1})\}$

in which on letting $n \rightarrow \infty$; using the continuity of S and (2. 1. 2), (2. 1. 11) and (2. 1. 13) we get

$$\lim_{n \rightarrow \infty} \mu(x_{2n}, Jx_{2n+1}) = \max \{S(z, Jz, Jz), S(z, z, z), S(Jz, Jz, Jz)\}$$

$$= S(z, Jz, Jz).$$

(b) When T is continuous, we have, by (2. 1. 2), that

(2. 1. 15) $T^2x_{2n+1} \rightarrow Tz$, $TJx_{2n+1} \rightarrow Tz$ as $n \rightarrow \infty$.

from (2. 1. 15) and (2. 1. 12), we get

(2. 1. 16) $JTx_{2n+1} \rightarrow Tz$ as $n \rightarrow \infty$.

Now we have

(2. 1. 17) $\mu(x_{2n}, Tx_{2n+1}) = \max \{S(Ix_{2n}, JTx_{2n+1}, JTx_{2n+1}), S(Ix_{2n}, Px_{2n}, Px_{2n}),$
 $S(JTx_{2n+1}, T^2x_{2n+1}, T^2x_{2n+1})\}$

In which on letting $n \rightarrow \infty$; using (2. 1. 2), (2. 1. 15), (2. 1. 16) and continuity of S we get $\lim_{n \rightarrow \infty} \mu(x_{2n}, Tx_{2n+1}) =$
 $\max \{S(z, Tz, Tz), S(z, z, z), S(Tz, Tz, Tz)\}$
 $= S(z, Tz, Tz).$

Hence the lemma.

2. 2 Prof of Theorem 2. 1: In this section we first prove the existence of a common fixed point in various cases of the condition (iii) in Theorem 2. 1.

Case(I). Suppose that I is continuous

Taking $x = Ix_{2n}$ and $y = x_{2n+1}$, in condition (ii) of the Theorem 2. 1, we have

(2. 2. 1) $S(PIx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq \Phi(\mu(Ix_{2n}, x_{2n+1}))$.

Also the continuity of S gives

$$S(Iz, z, z) = \lim_{n \rightarrow \infty} S(PIx_{2n}, Tx_{2n+1}, Tx_{2n+1})$$

Therefore, by Lemma 2. 1. 1, we get

$$(2. 2. 2) \quad S(Iz, z, z) = \lim_{n \rightarrow \infty} \sup S(PIx_{2n}, Tx_{2n+1}, Tx_{2n+1})$$

$$\leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(Ix_{2n}, x_{2n+1}))$$

$$= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(Ix_{2n}, x_{2n+1})))$$

$$= \Phi(\lim_{n \rightarrow \infty}(\mu(Ix_{2n}, x_{2n+1})))$$

$$= \Phi(S(Iz, z, z))$$

Thus

$$(2. 2. 3) \quad S(z, Iz, Iz) \leq \Phi(S(z, Iz, Iz))$$

Now we claim that $Iz = z$. In fact if $Iz \neq z$, $S(z, Iz, Iz) > 0$ so that

$\Phi(S(Iz, z, z)) < S(Iz, z, z)$, since Φ is contractive modulus, which contradicts (2. 2. 3). Therefore $Iz = z$.

Now the continuity of S gives

$$S(Pz, z, z) = \lim_{n \rightarrow \infty} S(Pz, Tx_{2n+1}, Tx_{2n+1})$$

$$= \lim_{n \rightarrow \infty} \sup S(Pz, Tx_{2n+1}, Tx_{2n+1})$$

using the condition (ii) of Theorem (2. 1) and upper semi continuity of Φ in the above we get

$$(2. 2. 4) \quad S(Pz, z, z) \leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(z, x_{2n+1}))$$

$$= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(z, x_{2n+1})))$$

$$= \Phi(\lim_{n \rightarrow \infty}(\mu(z, x_{2n+1})))$$

But

$$\lim_{n \rightarrow \infty} \mu(x_{2n}, x_{2n+1}) = \lim_{n \rightarrow \infty} \max \{S(Iz, Jx_{2n+1}, Jx_{2n+1}), S(Iz, Sz, Sz),$$

$$S(Jx_{2n+1}, Tx_{2n+1}, Tx_{2n+1})\}$$

$$= \max \{S(Iz, z, z), S(Iz, Pz, Pz), S(z, z, z)\}$$

since $Iz = z$, $Jx_{2n+1} \rightarrow z$ and $Tx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$. Therefore (2. 2. 4) gives

$$(2. 2. 5) \quad S(Pz, z, z) \leq \Phi(S(Pz, z, z))$$

Now we claim that $Pz = z$. In fact if $Pz \neq z$, $S(Pz, z, z) > 0$ and by the definition of Φ we get $\Phi(S(Pz, z, z)) < S(Pz, z, z)$, contradicting (2.2.5). Hence $Pz = z$.

Thus we have $Pz = Iz = z$.

Now, since $P(X) \subseteq J(X)$, there is a $u \in X$ with $z = Pz = Ju$, for some $u \in X$ and we have

$Pz = Iz = Ju = z$. We claim that $Tu = z$.

In fact if $Tu \neq z$, then $S(z, Tu, Tu) > 0$ and therefore by (ii) of Theorem 2.1 we get

$$\begin{aligned} S(z, Tu, Tu) &= S(Pz, Tu, Tu) \leq \Phi(\mu(z, u)) \\ &= \Phi(\max\{S(Iz, Ju, Ju), S(Iz, Pz, Pz), S(Ju, Tu, Tu)\}) \\ &= \Phi(S(z, Tu, Tu)). \end{aligned}$$

since $Iz = z = Pz = Ju$, and this implies

$S(z, Tu, Tu) \leq \Phi(S(z, Tu, Tu)) < S(z, Tu, Tu)$, which is a contradiction. Therefore $z = Tu$.

Hence we have $z = Pz = Iz = Ju = Tu$. Now taking $y_n = u$ for $n \geq 1$, it follows that $Ty_n \rightarrow Tu = z$ and $Jy_n \rightarrow Ju = z$ as $n \rightarrow \infty$. Also since the pair (T, J) is compatible we have

$\lim_{n \rightarrow \infty} S(TJy_n, JT y_n, JT y_n) = 0$, which gives $S(TJu, JT u, JT u) = 0$ or $TJu = JT u$ that is $Tz = Jz$. Now by condition (ii) of Theorem 2.1 we have

$$\begin{aligned} (2.2.6) \quad S(z, Tz, Tz) &= S(Pz, Tz, Tz) \leq \Phi(\mu(z, z)) \\ &= \Phi(\max\{S(Iz, Jz, Jz), S(Iz, Pz, Pz), S(Jz, Tz, Tz)\}) \\ &= \Phi(S(z, Tz, Tz)). \end{aligned}$$

since $Iz = z = Pz = Ju$ and $Jz = Tz$. Thus (2.2.6) gives

$S(z, Tz, Tz) \leq \Phi(S(z, Tz, Tz))$; which, as before, forces $Tz = z$.

Hence $z = Pz = Iz = Jz = Tz$. Thus z is a common fixed point of P, T, I and J .

Case (ii): Suppose that J is continuous

Taking $x = x_{2n}$ and $y = Jx_{2n+1}$, in condition (ii) of the Theorem 2.1, we have

(2.2.7) $S(Px_{2n}, TJx_{2n+1}, TJx_{2n+1}) \leq \Phi(\mu(x_{2n}, Jx_{2n+1}))$ in which letting $n \rightarrow \infty$, and using the facts $Px_{2n} \rightarrow z$ and $TJx_{2n+1} \rightarrow Jz$ as $n \rightarrow \infty$ (since the pair (T, J) is compatible and J is continuous), we get

$$\begin{aligned} (2.2.8) \quad S(z, Jz, Jz) &= \lim_{n \rightarrow \infty} S(Px_{2n}, TJx_{2n+1}, TJx_{2n+1}) \\ &= \lim_{n \rightarrow \infty} \sup S(Px_{2n}, TJx_{2n+1}, TJx_{2n+1}) \\ &\leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(x_{2n}, Jx_{2n+1})) \\ &= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(x_{2n}, Jx_{2n+1})) \\ &= \Phi(\lim_{n \rightarrow \infty}(\mu(x_{2n}, Jx_{2n+1})) \end{aligned}$$

Now, in view of (a) of Lemma 2.1.10, we get

$S(z, Jz, Jz) \leq \Phi(S(z, Jz, Jz))$, which leads to a contradiction if $Jz \neq z$. Therefore $Jz = z$.

Now the continuity of S gives

$$\begin{aligned} S(z, Tz, Tz) &= \lim_{n \rightarrow \infty} S(Px_{2n}, Tz, Tz) \\ &= \lim_{n \rightarrow \infty} \sup S(Px_{2n}, Tz, Tz) \end{aligned}$$

But, condition (ii) of Theorem (2.1) and upper semi continuity of Φ give

$$\begin{aligned} (2.2.9) \quad S(z, Tz, Tz) &\leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(x_{2n}, z)) \\ &= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(x_{2n}, z))) \\ &= \Phi(\lim_{n \rightarrow \infty}(\mu(x_{2n}, z))) \end{aligned}$$

Now since $\mu(x_{2n}, z) \rightarrow S(z, Tz, Tz)$ as $n \rightarrow \infty$ from condition (ii) of Theorem 2.1. and since Φ is a contractive modulus, (2.2.9) gives $S(z, Tz, Tz) \leq \Phi(S(z, Tz, Tz))$, from which we get a contradiction if $Tz \neq z$. Therefore $Tz = z$.

Hence we have

$$(2.2.10) \quad Tz = Jz = z.$$

Now, since $T(X) \subseteq I(X)$, there is a $u' \in X$ with $z = Tz = Iu'$,

Again by condition (ii) of Theorem 2.1 we get

$$(2.2.11) \quad S(Pu', z, z) = S(Pu', Tz, Tz) \leq \Phi(\mu(u', z))$$

$$= \Phi (\max \{S(Iu', Jz, Jz), S(Iu', Pu', Pu'), S(Jz, Tz, Tz)\}).$$

$$= \Phi (S(Su', z, z)).$$

since $Jz = Tz = z = Iu'$. That is

$S(Pu', z, z) \leq \Phi (S(Pu', z, z))$, which is a contradiction if $Pu' \neq z$. Therefore $z = Pu'$,

That is, $Pu' = Iu' = z$. Taking $y_n = u'$ for $n \geq 1$, we find that $Py_n \rightarrow Pu'$ and $Iy_n \rightarrow Iu'$ as $n \rightarrow \infty$. Now using the compatibility of the pair (P, I) we have

$$\lim_{n \rightarrow \infty} S(Py_n, IPy_n, IPy_n) = 0, \text{ which gives } S(PIu', IPu', IPu') = 0 \text{ or } PIu' = IPu'.$$

Hence $Sz = Iz$. Again by condition (ii) of Theorem 2. 1 we have

$$(2. 2. 12) S(Pz, z, z) = S(Pz, Tz, Tz) \leq \Phi(\mu(z, z)) = \Phi(S(Pz, z, z)).$$

$$\text{Since } \mu(z, z) = S(Pz, z, z).$$

Now (2. 2. 13) leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$. Hence $z = Pz = Iz = Jz = Tz$, showing that z is a common fixed point of P, T, I and J .

Case (iii): Suppose that P is continuous

Taking $x = Px_{2n}$ and $y = x_{2n+1}$, in condition (ii) of the Theorem 2. 1, we have

$$(2. 2. 13) S(P^2x_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq \Phi(\mu(Px_{2n}, x_{2n+1})).$$

Since $P^2x_{2n} \rightarrow Pz$ as $n \rightarrow \infty$ (by the continuity of P) $Tx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$, letting $n \rightarrow \infty$ in (2. 2. 13), we get

$$(2. 2. 14) S(Pz, z, z) = \lim_{n \rightarrow \infty} \sup S(P^2x_{2n}, Tx_{2n+1}, Tx_{2n+1})$$

$$= \lim_{n \rightarrow \infty} \sup S(P^2x_{2n}, Tx_{2n+1}, Tx_{2n+1})$$

$$\leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(Px_{2n}, x_{2n+1}))$$

$$= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(Sx_{2n}, x_{2n+1}))$$

$$= \Phi(\lim_{n \rightarrow \infty} (\mu(Px_{2n}, x_{2n+1})).$$

Now, in view of (b) of Lemma 2. 1. 1, (2. 2. 14) gives

$$S(Pz, z, z) \leq \Phi(S(Pz, z, z)), \text{ which is a contradiction if } Pz \neq z, \text{ by the definition of } \Phi. \text{ Therefore } Pz = z.$$

Now, since $P(X) \subseteq J(X)$, there is a $v \in X$ with $z = Pz = Jv$,

Now the continuity of S gives

$$S(z, Tv, Tv) = \lim_{n \rightarrow \infty} S(Px_{2n}, Tv, Tv)$$

$$= \lim_{n \rightarrow \infty} \sup S(Px_{2n}, Tv, Tv)$$

so that condition (ii) of Theorem 2. 1 and upper semi continuity of Φ give

$$(2. 2. 15) S(z, Tv, Tv) \leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(x_{2n}, v))$$

$$= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(x_{2n}, v)))$$

$$= \Phi(\lim_{n \rightarrow \infty} (\mu(x_{2n}, v)))$$

Since $\mu(x_{2n}, v) \rightarrow S(z, Tv, Tv)$, as $n \rightarrow \infty$ (from condition (ii) of Theorem 2. 1, (2. 2. 15) gives $S(z, Tv, Tv) \leq \Phi(S(z, Tv, Tv))$, from which we get a contradiction if $z \neq Tv$. Therefore $z = Tv$

Thus we have $Tv = Jv = z$.

Now taking $y_n = v$ for $n \geq 1$, we get $Ty_n \rightarrow Tv$ and $Jy_n \rightarrow Jv$ as $n \rightarrow \infty$. Also since the pair (T, J) compatibility we have

$$\lim_{n \rightarrow \infty} S(TJy_n, JT_y_n, JT_y_n) = 0, \text{ which gives } S(TJv, JTv, JTv) = 0 \text{ or } TJv = JTv, \text{ hence } Tz = Jz$$

Now the continuity of S gives

$$S(z, Tz, Tz) = \lim_{n \rightarrow \infty} S(Px_{2n}, Tz, Tz)$$

$$= \lim_{n \rightarrow \infty} \sup S(Px_{2n}, Tz, Tz)$$

But, condition (ii) of Theorem (2. 1) and upper semi continuity of Φ give

$$(2. 2. 16) S(z, Tz, Tz) \leq \lim_{n \rightarrow \infty} \sup \Phi(\mu(x_{2n}, z))$$

$$= \Phi(\lim_{n \rightarrow \infty} \sup(\mu(x_{2n}, z)))$$

$$= \Phi(\lim_{n \rightarrow \infty} (\mu(x_{2n}, z)))$$

Since $\mu(x_{2n}, z) \rightarrow S(z, Tz, Tz)$ as $n \rightarrow \infty$ from condition (ii) of Theorem 2.1, (2. 2. 16) gives $S(z, Tz, Tz) \leq \Phi(S(z, Tz, Tz))$, and this leads to a contradiction if $Tz \neq z$. Hence $Tz = z$.

Also since $T(X) \subseteq I(X)$ implies there is a $u'' \in X$ with $z = Tz = Iu''$. Now from condition(ii) of Theorem 2. 1, we have

(2. 2. 17) $S(Pu'', z, z) = S(Pu'', Tz, Tz) \leq \Phi(\mu(u'', z))$

Since $\mu(u'', z) = S(Pu'', Tz, Tz)$, from (ii)' of Theorem 2. 1, (2. 2. 17) gives

$S(Pu'', z, z) \leq \Phi(S(Pu'', z, z))$, and this will be a contradiction if $Pu'' \neq z$. Therefore $z = Pu''$. Hence $Pu'' = Iu'' = z$. Also, since the pair (P, I) is compatible, we have $\lim_{n \rightarrow \infty} S(PIy_n, IPy_n, IPy_n) = 0$.

Taking $y_n = u''$ for $n \geq 1$, we get $S(PIu'', IPu'', IPu'') = 0$. Therefore $PIu'' = IPu''$ or $Pz = Iz$. Hence $z = Pz = Iz = Jz = Tz$, showing that z is a common fixed point of P, T, I and J .

Case(iv): Finally suppose that T is continuous

Taking $x = x_{2n}$ and $y = Tx_{2n+1}$, in condition (ii) of the Theorem 2. 1, we have

(2. 2. 18) $S(Px_{2n}, T^2x_{2n+1}, T^2x_{2n+1}) \leq \Phi(\mu(x_{2n}, Tx_{2n+1}))$.

Since $Px_{2n} \rightarrow z$ and $T^2x_{2n+1} \rightarrow Tz$ as $n \rightarrow \infty$ (by the continuity of T) letting $n \rightarrow \infty$ in (2. 2.18) we get

$$\begin{aligned} (2. 2. 19) \quad S(z, Tz, Tz) &= \limsup_{n \rightarrow \infty} S(Px_{2n}, T^2x_{2n+1}, T^2x_{2n+1}) \\ &= \limsup_{n \rightarrow \infty} S(Px_{2n}, T^2x_{2n+1}, T^2x_{2n+1}) \\ &\leq \limsup_{n \rightarrow \infty} \Phi(\mu(x_{2n}, Tx_{2n+1})) \\ &= \Phi(\limsup_{n \rightarrow \infty} (\mu(x_{2n}, Tx_{2n+1}))) \\ &= \Phi(\lim_{n \rightarrow \infty} (\mu(x_{2n}, Tx_{2n+1}))). \end{aligned}$$

Which, in view of (b) of Lemma 2. 1. 1, (2. 2. 10) gives

$S(z, Tz, Tz) \leq \Phi(S(z, Tz, Tz))$, and this will be contradiction if $Tz \neq z$, by the definition of Φ . Therefore $Tz = z$.

Now, since $T(X) \subseteq I(X)$, there is a $v' \in X$ with $z = Tz = Iv'$,

Now the continuity of S gives

$$\begin{aligned} S(Pv', z, z) &= \lim_{n \rightarrow \infty} S(Pv', Tx_{2n+1}, Tx_{2n+1}) \\ &= \lim_{n \rightarrow \infty} \sup S(Pv', Tx_{2n+1}, Tx_{2n+1}) \end{aligned}$$

Now condition (ii) of Theorem 2. 1 and upper semi continuity of Φ give

$$\begin{aligned} (2. 2. 20) \quad S(Pv', z, z) &\leq \limsup_{n \rightarrow \infty} \Phi(\mu(v', x_{2n+1})) \\ &= \Phi(\limsup_{n \rightarrow \infty} (\mu(v', x_{2n+1}))) \\ &= \Phi(\lim_{n \rightarrow \infty} (\mu(v', x_{2n+1}))) \end{aligned}$$

Since $\mu(v', x_{2n+1}) \rightarrow S(Pv', z, z)$, as $n \rightarrow \infty$ (from condition (ii) of Theorem 2. 1), (2. 2. 20) gives $S(Sv', z, z) \leq \Phi(S(Pv', z, z))$, and this leads to a contradiction if $Sv' \neq z$. Therefore $Sv' = z$.

Hence $Pv' = Iv' = z$.

Since the pair (P, I) is compatibility, we have $\lim_{n \rightarrow \infty} S(PIy_n, IPy_n, IPy_n) = 0$.

Taking $y_n = v'$ for $n \geq 1$, we get $S(PIv', IPv', IPv') = 0$ (since $Py_n \rightarrow Pv', Iy_n \rightarrow Iv'$ as $n \rightarrow \infty$). Therefore $PIv' = IPv'$ or $Pz = Tz$.

Now the continuity of S gives

$$\begin{aligned} S(Pz, z, z) &= \lim_{n \rightarrow \infty} S(Pz, Tx_{2n+1}, Tx_{2n+1}) \\ &= \lim_{n \rightarrow \infty} \sup S(Pz, Tx_{2n+1}, Tx_{2n+1}) \end{aligned}$$

Again, by condition (ii) of Theorem 2. 1 and upper semi continuity of Φ gives

$$\begin{aligned} (2. 2. 21) \quad S(Pz, z, z) &\leq \limsup_{n \rightarrow \infty} \Phi(\mu(z, x_{2n+1})) \\ &= \Phi(\limsup_{n \rightarrow \infty} (\mu(z, x_{2n+1}))) \\ &= \Phi(\lim_{n \rightarrow \infty} (\mu(z, x_{2n+1}))) \end{aligned}$$

Since $\mu(z, x_{2n+1}) \rightarrow S(Pz, z, z)$ as $n \rightarrow \infty$ from condition (ii) of Theorem 2.1, (2. 2. 21) gives $S(Pz, z, z) \leq \Phi(S(Pz, z, z))$, and this leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$. Hence $Pz = Iz = Tz = z$

Now since $P(X) \subseteq J(X)$, this implies there is a $v'' \in X$ with $z = Pz = Jv''$. Now from condition (ii) of Theorem 2. 1, we have

(2. 2. 22) $S(z, Tv'', Tv'') = S(Pz, Tv'', Tv'') \leq \Phi(\mu(z, v''))$

Since $\mu(z, v'') = S(z, Tv'', Tv'')$, from (ii)' of Theorem 2. 1, (2. 2. 22) gives

$S(z, Tv'', Tv'') \leq \Phi(S(z, Tv'', Tv''))$, and this will be a contradiction if $Tv'' \neq z$. Therefore $z = Tv''$. Hence $Tv'' = Jv'' = z$. Also, since the pair (T, J) is compatible, we have $\lim_{n \rightarrow \infty} S(TJy_n, JTy_n, JTy_n) = 0$. Taking $y_n = v''$ for $n \geq 1$, we get $S(TJv'', JTv'', JTv'') = 0$. Therefore $TJv'' = JTv''$ or $Tz = Jz$. Therefore $z = Pz = Iz = Jz = Tz$. Hence z is a common fixed point of P, T, I and J .

Now we prove the **uniqueness** of the common fixed point

If possible let z' be another common fixed point of S, T, I and J . Then from condition (ii) of Theorem 2. 1, we have

$$(2. 2. 23) \quad S(z, z', z') = S(Pz, Tz', Tz') \leq \Phi(\mu(z, z')).$$

Since $\mu(z, z') = S(z, z', z')$ from (ii)' of Theorem 2. 1, (2. 2. 23) gives

$S(z, z', z') \leq \Phi(S(z, z', z'))$ and this will be contradiction if $z \neq z'$. Therefore $z = z'$. Thus z is the unique common fixed point of P, T, I and J .

To prove that z is the unique common fixed point of P and I ; and of T and J . Let w be another common fixed point of P and I . Then $z = Pz = Tz = Iz = Jz$ and $w = Iw = Pw$.

Now, from condition (ii) of Theorem 2. 1, we have

$$(2. 2. 24) \quad S(z, w, w) = S(w, z, z) = S(Pw, Tz, Tz) \leq \Phi(\mu(w, z)).$$

Since $\mu(w, z) = S(w, z, z)$ (from (ii)' of Theorem 2. 1), (2. 2. 24) gives

$S(w, z, z) \leq \Phi(S(w, z, z))$, and this will be a contradiction if $w \neq z$. Therefore $w = z$. Thus z is the unique common fixed point of P and I . Similarly we can prove that z is the unique common fixed point of T and J . Thus the Theorem 2. 1 is completely proved.

2.3 A Common Fixed Point Theorem for Four Selfmaps of a Complete S- metric space:

Before we prove the main result of this section, we prove the following lemma:

2.3.1 Lemma: Let (X, S) be a S- metric space and P, T, I and J be selfmaps of X such that

(i) $P(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$

(ii) $S(Px, Ty, Ty) \leq c \cdot \mu(x, y)$ for all $x, y \in X$

where $0 \leq c < 1$ and

$$\mu(x, y) = \max \{S(Ix, Jy, Jy), S(Ix, Px, Px), S(Jy, Ty, Ty)\}$$

Further, if

(i) (X, S) is complete.

Then for any $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the four selfmaps, such that $Px_0, Tx_1, Px_2, Tx_3, \dots, Px_{2n}, Tx_{2n+1}, \dots$ converges to some $z \in X$.

Proof: Suppose P, T, I and J are selfmaps of a S-metric space (X, S) for which the conditions (i) and (ii) hold.

Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to four selfmaps. Then, since $Px_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n \geq 0$. Note that

$$\begin{aligned} \mu(x_{2n}, x_{2n+1}) &= \max \{S(Ix_{2n}, Jx_{2n+1}, Jx_{2n+1}), S(Ix_{2n}, Px_{2n}, Px_{2n}), S(Jx_{2n}, Tx_{2n+1}, Tx_{2n+1})\} \\ &= \max \{S(Ix_{2n}, Px_{2n}, Px_{2n}), S(Ix_{2n}, Px_{2n}, Px_{2n}), S(Jx_{2n+1}, Tx_{2n+1}, Tx_{2n+1})\} \\ &= \max \{S(Ix_{2n}, Px_{2n}, Px_{2n}), S(Jx_{2n+1}, Tx_{2n+1}, Tx_{2n+1})\} \end{aligned}$$

$$\mu(x_{2n}, x_{2n+1}) = \max \{S(Tx_{2n-1}, Px_{2n}, Px_{2n}), S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1})\}$$

This together with (ii) of Theorem 2. 1 gives

$$\begin{aligned} S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1}) &\leq c \cdot \mu(x_{2n}, x_{2n+1}) \\ &\leq c \cdot \max \{S(Tx_{2n-1}, Px_{2n}, Px_{2n}), S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1})\}. \end{aligned}$$

Since $0 \leq c < 1$, it follows from the above inequality that

$$\max \{S(Tx_{2n-1}, Px_{2n}, Px_{2n}), S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1})\} = S(Tx_{2n-1}, Px_{2n}, Px_{2n})$$

$$\text{Therefore } S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq c \cdot S(Tx_{2n-1}, Px_{2n}, Px_{2n}) \dots\dots\dots (A)$$

Similarly, we can prove

$$S(Tx_{2n-1}, Px_{2n}, Px_{2n}) \leq c \cdot S(Px_{2n-2}, Tx_{2n-1}, Tx_{2n-1}) \dots\dots\dots (B)$$

From (A) and (B), we get

$$\begin{aligned} S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1}) &\leq c^2 S(Px_{2n-2}, Tx_{2n-1}, Tx_{2n-1}) \\ &\leq c^4 S(Px_{2n-4}, Tx_{2n-3}, Tx_{2n-3}) \\ &\dots\dots\dots \end{aligned}$$

$$\leq c^{2n} S(Px_0, Tx_1, Tx_1) \rightarrow 0$$

Since $c^{2n} \rightarrow 0$ as $n \rightarrow \infty$ (because $c < 1$). Hence the sequence $Px_0, Tx_1, Px_2, Tx_3, \dots, Px_{2n}, Tx_{2n+1}$ is a Cauchy sequence in (X, S) and since it is complete, it converges to a point say $z \in X$, proving lemma.

2.3.2 Remark: The converse of lemma is not true. That is, suppose P, T, I and J are selfmaps of a S -metric space (X, S) satisfying condition (i) and (ii) of Lemma 2.3.2. Even, if for each $x_0 \in X$ and for each associated sequence $\{x_n\}$ of x_0 relative to P, T, I and J the sequence $Px_0, Tx_1, Px_2, Tx_3, \dots, Px_{2n}, Tx_{2n+1}, \dots$ converges in X , then (X, S) need not be complete.

2.3.3 Theorem: Suppose (X, S) is a S -metric space satisfying conditions (i) to (iv) of Theorem 2.1. Further, if $(v)'$ (X, S) is complete then P, T, I and J have a unique common fixed point $z \in X$. Also z is the unique common fixed point for the pair P and I ; and for the pair T and J .

Proof: In view of Lemma 2.3.1 the condition (v) of Theorem 2.1 holds; because of $(v)'$. Hence the Theorem follows from Theorem 2.1.

2.3.4 Corollary ([5]): Let P, T, I and J be selfmaps of a metric space (X, d) satisfying the conditions:

- (i) $P(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$
- (ii) $d(Px, Ty) \leq c \mu_0(x, y)$ for all $x, y \in X$, where $0 \leq c < 1$ and
- (ii)' $\mu_0(x, y) = \max \{d(Ix, Jy), d(Ix, Px), d(Jy, Ty)\}$ for $x, y \in X$
- (iii) one P, T, I and J is continuous, and
- (iv) $PI = IP$ and $TJ = JT$
- Further, if
- (v) X is complete

Then the four selfmaps P, T, I and J have a unique common fixed point in $z \in X$. Further z is the unique common fixed point of P and I ; and of T and J .

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If $S(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$ then (X, S) is a S -metric space and $S(x, y, x) = d(x, y)$. Therefore condition (ii) can be written as $S(Px, Ty, Ty) \leq c \cdot \mu(x, y)$ for all $x, y \in X$ where $\mu(x, y) = \max \{S(Ix, Jy, Jy), S(Ix, Px, Px), S(Jy, Ty, Ty)\}$ which is the same as condition (ii) of Theorem 2.3.3. Also since (X, d) is complete, we have (X, S) is complete by Corollary 1.14.

Now P, T, I and J are selfmaps on (X, S) satisfying conditions of Theorem 2.3.3 and hence the corollary follows.

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